A Discrete Velocity Model for Relaxation of Anisotropic Distribution Functions.

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Abstract

A discrete velocity model is employed to simulate the nonlinear relaxation of anisotropic distribution functions. As a test of the simulation, the relaxation of a nonequilibrium distribution function of a test-particle dilutely dispersed in a second major constituent is considered. The results with the discrete velocity model are compared with expansion solutions of the Boltzmann equation. The relaxation of anisotropic distributions in non-neutral plasmas is also considered.

1 Introduction

The purpose of this paper is to study the applicability of a discrete velocity model to the relaxation of an anisotropic non-neutral plasma for which the relaxation is due to electron-electron Coulomb collisions. An alternate approach used in plasma physics is based on solutions of the nonlinear Fokker–Planck equation [1]. One of the main difficulties is the singularity of the differential Rutherford cross section at small scattering angles which requires the use of a cut-off in angle or impact parameter, or the more rigrous Lenard-Balescu equation [2]. The present paper considers the integral form of the nonlinear Boltzmann equation and the representation in the discrete velocity approximation. The methodology that we use follows the ones reported recently [3]-[5]. The main objective is to consider the relaxation to equilibrium of an initial nonequilibrium anisotropic electron distribution function. The initial distribution is taken to be a bi-Maxwellian with different temperatures characterizing the transverse and longitudinal translational energies, analogous to the conditions of recent experimental studies [6].

As a test of the simulation, we consider the relaxation of a neutral constituent dilutely dispersed in a large excess of a second component which remains at equilibrium. For the hard sphere interaction between the two components, the relaxation of the 'test-particle' distribution function can be calculated with the expansion of the distribution function in Sonine polynomials. We compare these solutions and those obtained with the discrete velocity model in terms of the relaxation of the temperature of the test-particles.

Section 2 of the paper briefly discusses the details of the discrete velocity model that is used. Section 3 presents the benchmark test case. The results for the nonlinear relaxation of anisotropic non-neutral plasmas is presented and discussed in Section 4.

2 Discrete Velocity Model

The Boltzmann equation for this problem is

$$\frac{\partial f(\mathbf{v}_1, t)}{\partial t} = \iint [f_1' f' - f_1 f] \, g \, \sigma(g, \chi) \, d\chi d\mathbf{v}. \tag{2.1}$$

The Boltzmann equation is strictly valid for a gas at low densities for which only short range binary collisions are important. A common assumption in plasma physics is that large angle collisions are unimportant relative to the more common small angle collisions. The effects of large impact parameter collisions are often discarded by cutting off the cross-section at a collision angle corresponding to an impact parameter equal to the Debye length. The Boltzmann equation is thus transformed to the differential Fokker–Planck equation. Shoub [7,8] has recently questioned this approximation for high energy test particles in a Maxwellian bath.

In the present paper, we proceed from the integral form of the Boltzmann equation and apply the discrete velocity model so that eqn (2.1) is replaced by the discrete Boltzmann equation (DBE) given by,

$$\begin{aligned} \frac{dN_i}{dt} &= \sum_{j,k,l=1}^{p} A_{ij}^{kl}(N_k N_l - N_i N_j), \quad i = 1, ..., p = (2r)^3 \\ &= \sum_j \left[\left(\sum_{k,l} A_{ij}^{kl} N_k N_l \right) - \left(\sum_{k,l} A_{ij}^{kl} \right) N_i N_j \right], \end{aligned}$$

where the $N_i(t)$ are values of the distribution function on the box centered on velocity i, and

$$A_{ij}^{kl} = |v_i - v_j| a_{ij}^{kl}.$$

Here A_{ij}^{kl} is the transition probability for the collision pair with initial velocities v_i, v_j resulting in v_k, v_l velocities. The effect of the cross-section is given by a_{ij}^{kl}, the probability of choosing the k, l outcome from a i, j collision. The resolution of the velocity grid is specified by, r, the number of speeds on one half-axis.

The discrete Boltzmann equation can be written

$$\frac{d\mathbf{N}(t)}{dt} = L(\mathbf{N}) = C(\mathbf{N}) - D(\mathbf{N}), \tag{2.2}$$

where $\mathbf{N}(t) = (N_1(t), N_2(t), \ldots, N_p(t))$ is the discrete representation of the distribution function, $C(\mathbf{N})$ is the gain term in the DBE, and $D(\mathbf{N})$ is the loss term. A Taylor expansion of $\mathbf{N}(t)$ in t is

$$\mathbf{N}^{i+1} = \mathbf{N}^i + \Delta t \frac{d\mathbf{N}^i}{dt}\bigg|_i + \frac{1}{2}(\Delta t)^2 \frac{d^2\mathbf{N}^i}{dt^2}\bigg|_i + O[(\Delta t)^3], \tag{2.3}$$

where the abbreviations $\mathbf{N}^i = \mathbf{N}(t_i)$ and $\mathbf{N}^{i+1} = \mathbf{N}(t_i + \Delta t)$ are used. The derivative terms can be rewritten as $d\mathbf{N}^i/dt = L^i$ and $d^2\mathbf{N}^i/dt^2 = (L^i - L^{i-1})/\Delta t$, where $L^i = L(\mathbf{N}^i)$. This gives a second order (in Δt) explicit scheme for N,

$$N^{i+1} = N^i + \frac{\Delta t}{2}(3L^i - L^{i-1}) + O[(\Delta t)^3]. \tag{2.4}$$

An implicit scheme can be obtained by combining forward and backward time steps,

$$\begin{aligned} N^{i+1/2} &= N^i + \frac{\Delta t}{2}\frac{dN}{dt}\bigg|_i + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{d^2N}{dt^2}\bigg|_i + O[(\Delta t)^3], \\ &= N^{i+1} - \frac{\Delta t}{2}\frac{dN}{dt}\bigg|_{i+1} + \frac{1}{2}\left(\frac{-\Delta t}{2}\right)^2 \frac{d^2N}{dt^2}\bigg|_{i+1} + O[(\Delta t)^3]. \end{aligned}$$

to give the second order implicit scheme,

$$N^{i+1} = N^i + \frac{\Delta t}{2}(L^{i+1} + L^i) + O[(\Delta t)^3].$$

The C term is adequately treated with the explicit method, but the D term requires the implicit scheme to be stable. Approximate values for D^{i+1} are computed with an iteration. The estimate $D^{i+1} = D^i$ is used to compute provisional values for $\mathbf{N}^{i+1}$. Then improved estimates for D^{i+1} are computed and $\mathbf{N}^{i+1}$ is re-evaluated. In the linear test particle problem the same scheme is used but no iteration is needed for the D term.

3 Benchmark Test Problem

In this section, we consider the relaxation of a test-particle distribution in a large excess of a second component. The initial distribution of the minor species is a Maxwellian at a temperature different from the temperature of the major species which is assumed to remain at equilibrium. If the distribution function is written as, $f_1(v,t) = f_1^{(0)}(v)[1+\phi(v,t)]$, then the Boltzmann equation for the perturbation $\phi(v,t)$ is given by,

$$f_1^{(0)}\frac{\partial \phi}{\partial t} = \int f_1^{(0)}(v_1) f_2^{(0)}(v_2)[\phi(v_1',t) - \phi(v_1,t)]\sigma(g,\chi)\, g\, d\chi d\mathbf{v}_2. \quad (3.1)$$

We expand ϕ in Sonine polynomials, $\phi(x,t) = \sum_{n=1}^{\infty} a_n(t) S_n(x^2)$ where $x = \sqrt{mc^2/2kT}$ is the reduced speed. We reduce the Boltzmann equation to a set of linear differential equations, $da_m/dt = \sum_{n=1}^{\infty} A_{mn} a_n$ where A_{mn} are the matrix elements of the collision operator. The solution is of the form, $a_n(t) = \sum_{k=0}^{N} c_{nk} e^{-\lambda_k t}$, where the c_{nk} coefficients are chosen to match the intial condition, and λ_k are the eigenvalues of the collision operator. The temperature of the minor species is given by $T_1(t) = T_2[1 - a_1(t)]$. Figure 1 shows a comparison for the hard sphere cross section between the results obtained with the discrete velocity model and the results with the expansion in Sonine polynomials. Excellent agreement is obtained. The computational burden for the DBE is high, although the polynomial method may be difficult to apply for other cross sections and diverges for $T_2/T_1(0) \leq 0.5$. With this test calculation, the DBE code has been verified.

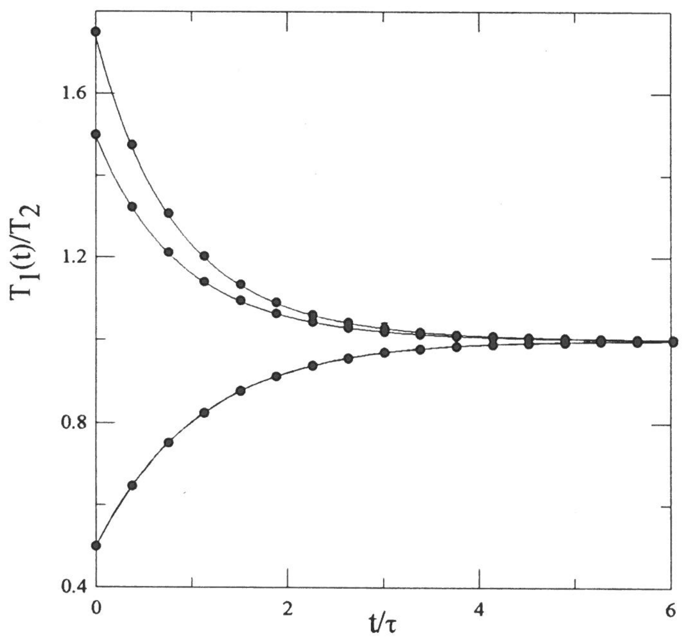

Figure 1. Temperature relaxation of a minor species. Solid lines are the polynomial solutions (N=7) and the symbols are the results with a discrete velocity model (r=7); hard sphere cross section; $\frac{1}{\tau} = n_2\sqrt{2kT_2/m_1}\pi d^2$; $m_1 = m_2$.

4 Relaxation of Anisotropic Distributions

The relaxation of an initial anisotropic distribution for a single species was considered with the discrete velocity model. The initial distribution function was taken to be an anisotropic bi-Maxwellian distribution of the form,

$$f^{Bi-Max}(\mathbf{v}) = n\sqrt{\frac{m}{2\pi kT_{\|}}\frac{m}{2\pi kT_{\perp}}}\exp[-\frac{m(v_x^2+v_y^2)}{2kT_{\perp}} - \frac{mv_z^2}{2kT_{\|}}] \tag{4.1}$$

A measure of the degree of the anisotropy is given by $\alpha(t) = T_{\|}(t)/T_{\perp}(t)$.

Figure 2 shows the relaxation of an anisotropic distribution function in terms of $\alpha(t)$ as obtained with the discrete velocity model for a hard sphere cross section. Three different numbers of discrete velocities corresponding to r=6, 7 and 8 were employed, with results that are essentially indistinguishable to the resolution of the graph.

The Coulomb cross section is given by,

$$\sigma(g,\chi) = \frac{e^4}{m^2g^4}\sin^{-4}(\frac{\chi}{2}). \tag{4.2}$$

As mentioned previously, the collision operator in the Boltzmann

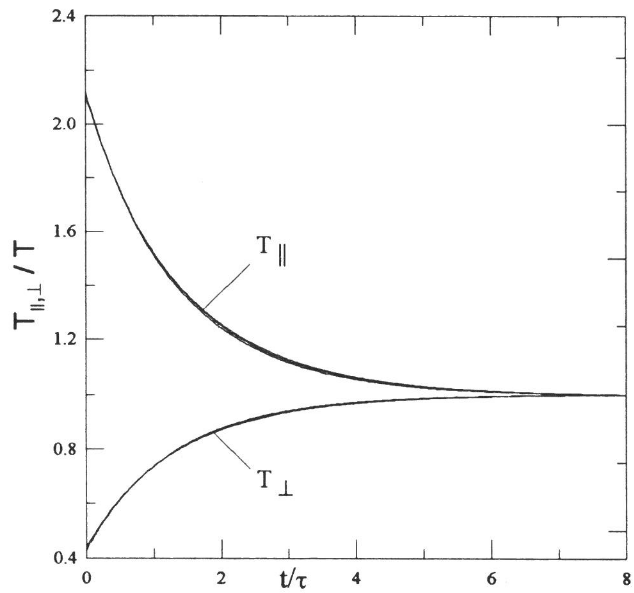

Figure 2. Relaxation of an anisotropic distribution function. Convergence of the discrete velocity model for a hard sphere cross section. The curves for resolutions r= 6, 7 and are 8 barely distinguishable, $\frac{1}{\tau} = n_2\sqrt{2kT_2/m_1}\pi d_c^2$.

equation is undefined because the differential cross section diverges at small scattering angles. The objective is to develop a discrete velocity model for Coulomb collisions. The singularity in the differential cross section at small scattering angles poses the main difficulty. In this preliminary application of a discrete velocity model, we have considered two model cross sections derived from the Rutherford cross section, eqn (4.2).

The relaxation for an anisotropic distribution in a nonneutral plasma is shown in Figure 3. Figure 3(A) is for the model cross section which is anisotropic but independent of energy, $\sigma(g,\chi) \approx \sin^{-4}(\chi/2)$. Figure 3(B) is for the isotropic cross section, $\sigma(g,\chi) \approx g^{-4}$. For both the resolution in the discrete velocity model is high, r=10 in Figure 3(A) and r=7 in Figure 3(B). The present version of the discrete velocity code was not sufficiently stable to provide results for the realistic Coulomb cross section, eqn (4.2). Work is in progress to modify the code appropriately. The present work complements the recent work of Shoub [8] in a critique of the use of the Fokker–Planck equation, as well as earlier work by Przybylski and Ligou [9] and Pomraning [10] on modification of the Fokker–Planck equation to take account of small impact parameters.

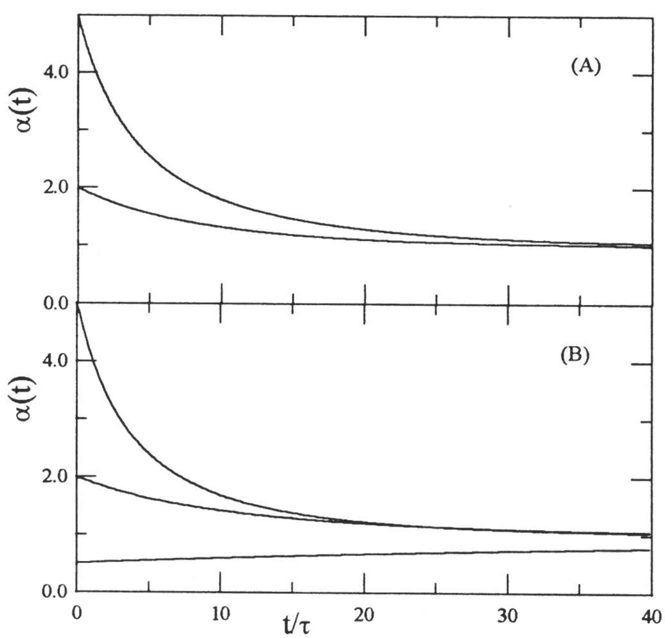

Figure 3. Relaxation of an anisotropic distribution function with the discrete velocity model, $\alpha(t) = T_{\|}/T_{\perp}$. (A) anisotropic cross section model, $\sigma(g,\chi) \approx \sin^{-4}(\chi/2)$, r =10; (B) isotropic Coulomb cross section, $\sigma(g,\chi) \approx g^{-4}$, r = 10.

5 Summary

A discrete velocity code was developed for the study of the relaxation of anisotropic distributions in nonneutral plasmas. The computational burden is high as a large number of discrete velocities are required to achieve convergence. The code was successfully benchmarked against known solutions for a linear relaxation problem. It provided accurate solutions to the Boltzmann equation for systems with short ranged potentials and for model Coulomb cross sections. Work is underway to modify the code so as to be applicable to realistic plasma systems.

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